



Conway and Aperiodic Tilings

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John Conway became interested in the kite and dart tilings (Figure 1) soon after their discovery by Roger Penrose around 1976 and was well known for his analysis of them and the memorable terminology he introduced. (John enhanced many subjects with colorful terminology, probably every subject on which he spent much time.) Neither Penrose nor Conway actually published much about the tilings; word spread initially through talks given by John, at least until 1977, when Martin Gardner wrote a wonderful “Mathematical Games” article in *Scientific American* [7], tracing their history back to mathematically simpler tilings by Robert Berger, Raphael Robinson, and others. I stumbled on the Gardner article in 1983 and became interested in the earlier tilings on realizing they could be used to make unusual models of solids in statistical physics.

I worked on this connection for a number of years, and when Conway was visiting the knot theory group in Austin in 1991, I asked him about a technical point on which I had been thinking. Within two hours he answered my question, and for the next 10 years or so we had frequent correspondence, which among other things resulted in three papers. It was great fun working with John, and I learned a lot, in part due to his unusual approach to mathematical research. I will describe our interaction and the results that came out of it in that decade, but to place the work in context, I will start with the prehistory as I first learned it from Gardner’s article.

The Origins of Aperiodicity

In 1961, the logician Hao Wang considered the following structure. Start with a finite number of unit squares in the plane, and deform their edges with small bumps and dents [11]. Wang was interested in tilings of the plane by translations of these “tiles,” a tiling being a covering without overlap. (Think of the squarish tiles in a tiling fitting together as in an infinite jigsaw puzzle, bumps fitting into neighboring dents and with corners touching and producing a copy of \mathbb{Z}^2 .)

Wang asked whether there exists such a finite collection of tiles (i.e., of edge deformations of unit squares) with these two properties:

- (a) One can tile the plane by translates of the tiles.
- (b) No such tiling is periodic, that is, invariant under a pair of linearly independent translations.

Intuitively, he was asking whether there is a finite collection of deformed squares such that translation copies can tile the plane but only in complicated ways, namely aperiodically. (Other notions of “complicated,” such as “algorithmically complex,” were considered by others later, but this is the original, brilliant, idea.)

Wang was interested in this question because the solution would answer an open decidability problem in predicate calculus, an indication of its significance. His student Robert Berger finally solved the tiling problem in the affirmative in his 1966 thesis [1], though Wang had already solved the logic problem another way. Berger found a number of such aperiodic tiling sets, well motivated but all rather complicated. Over the years, nicer solutions were found, in particular by Robinson, when the example by Penrose appeared in Gardner’s column in 1977: two (nonsquare) shapes, a kite and a dart, with their edges deformed (not shown in the tiling), appear in Figure 2.

This example doesn’t quite fit Wang’s structure, though the difficulties are easily resolved. The fact that the tiles are not square isn’t so important, since they are still compatible with nonorthogonal axes. And since the tiles appear in a tiling in only 10 different orientations, one could simply call rotated tiles different tile types and eschew rotations. In any case, the tilings were certainly interesting to many readers of Gardner’s column.

Wang’s invention of aperiodicity evolved in several directions, including physics modeling, which was the source of my original interest. This article celebrates John Conway and his mathematics, and I will concentrate on the geometric developments arising from the kite and dart tilings rather than the broader subject.

The Pinwheel and Quaquaversal Tilings

When John visited Austin in 1991, I pointed out the curious fact that in all known aperiodic tiling sets, for instance the kite and dart, the tiles appear in only a finite number of relative orientations. By analogy to particle configurations, this seemed artificial, and I thought there ought to be examples without that special feature. On the face of it, this was a hard problem—new aperiodic tiling sets are hard to find, even without this new property—but there was a natural path that I thought might yield such an example if only one could solve a certain much simpler problem first. I asked John whether there exists a polygon in the plane that can be decomposed into smaller copies of itself, all the

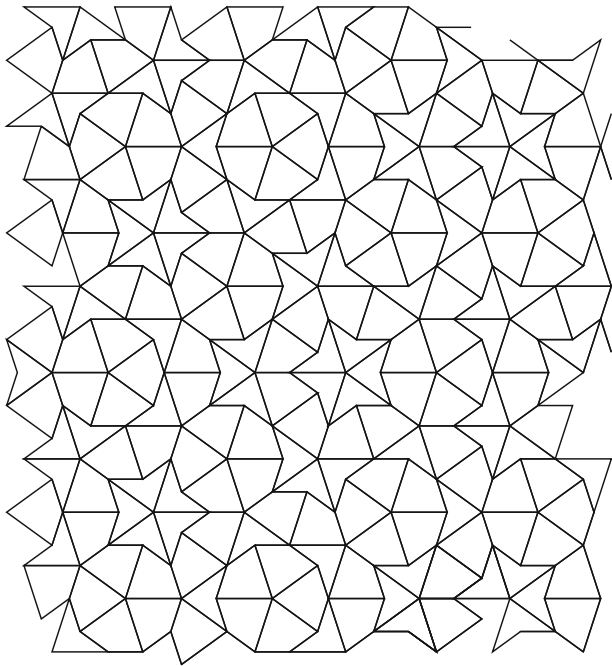


Figure 1. A kite and dart tiling.

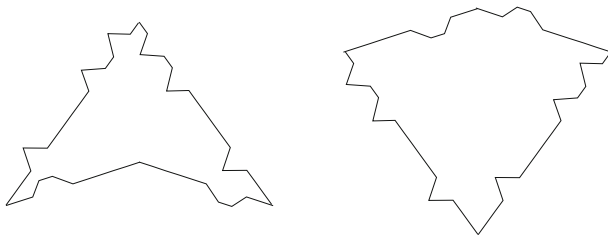


Figure 2. Kite and dart tiles with edge deformations to force aperiodicity.

same size, such that their relative orientations include an angle irrational with respect to π . He quickly came up with the $1, 2, \sqrt{5}$ right triangle with the decomposition shown in Figure 3. This did not solve my aperiodic tiling problem, but I hoped that I could now solve it using a beautiful recent result due to Shahar Mozes concerning tilings with squarish tiles. In his 1989 master's thesis [8], Mozes had studied the technique of Berger and Robinson and broadly generalized it. I will give one ingredient in Mozes's theorem through an example.

A simple method for building a complicated sequence of symbols is by substitution. For instance, the Thue–Morse sequence

$$abbabaabbaababba\dots$$

can be generated by repeating the substitution $a \rightarrow ab$, $b \rightarrow ba$, obtaining consecutively

$$a \rightarrow ab \rightarrow abba \rightarrow abbabaab \rightarrow \dots$$

On applying this idea to arrays in the plane, the substitution in Figure 4 gives rise to the infinite array in Figure 5.

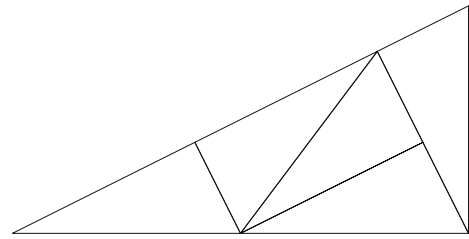


Figure 3. Pinwheel substitution.

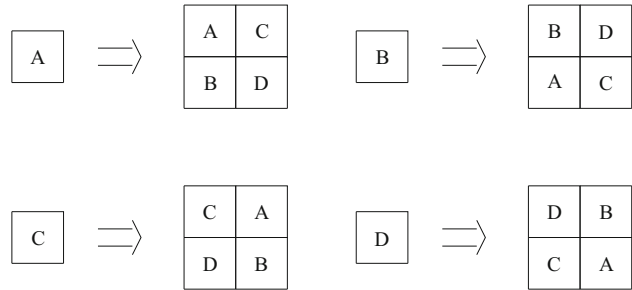


Figure 4. Morse substitution.

C	A	A	C	C	A	C
C	A	A	C	C	A	C
D	B	B	D	D	B	D
D	B	B	D	D	B	D
C	A	A	C	C	A	C
D	B	B	D	D	B	D
C	A	A	C	C	A	C

Figure 5. Morse tiling.

Mozes used substitutions to generalize Robinson's technique of adding bumps and dents to the edges of the unit squares so that the tilings by the jigsaw pieces are precise analogues of those in the infinite substitutions. (In his language, he was studying subshifts with \mathbb{Z}^2 action, and he proved how, given a substitution subshift, to construct a subshift of finite type that is conjugate to it as a dynamical system.) It should be noted that this process usually produces many differently deformed copies of each of the original square shapes.

Iterating the pinwheel substitution of Figure 3 by decomposing each triangle as indicated and then expanding the whole collection by a factor $\sqrt{5}$ about some point

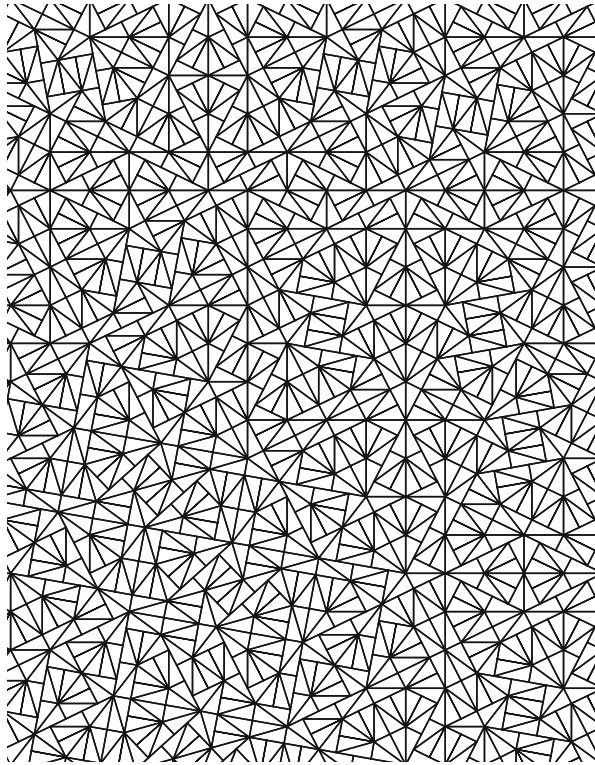


Figure 6. A pinwheel tiling.

and repeating leads to substitution pinwheel tilings such as shown in Figure 6. They will fill the plane if one uses appropriate expansion points, and they will have the desired range of orientations, since the angle $\arctan(\frac{1}{2})$ is irrational with respect to π .

Before going further, I should mention a complication, in that if I succeeded in finding deformed triangles that tile the plane only as in Figure 6, it would require infinitely many different tiles if one used only translations as in (a). What I was hoping to do was find a finite number of tiles satisfying the following two conditions:

- (a') Congruent copies can tile the plane.
- (b) No such tiling is periodic.

Note that geometry is slipping in by the replacement of the translation symmetries of \mathbb{Z}^2 by congruences of the plane in condition (a'), a change that is not in the spirit of Wang's invention, which was purely algorithmic, but which opened the door to interesting geometry, as we will see.

The square grid is heavily used by Mozes (and Berger and Robinson), so it was not at all clear that the technique could be adapted when the triangular tiles are also rotated as in the substitution of Figure 3. But I eventually adapted it in 1992, though the proof is much more complicated than that of Mozes. This gave a large but finite set of differently deformed triangular tiles that tile only as in Figure 6.

John was not interested in this. I worked with him for years, and it was clear from the beginning that he was very particular as to what he was willing to think about. What attracted him to the kite and dart tilings was the challenge,

which motivated Penrose also, to find an aperiodic tiling set with as few tiles as possible. He loved puzzles, in particular those involving geometry in the sense of Euclid. He loved discrete math in various senses. What he did not love was analysis, and there was no way I could get him interested in the ergodic theory of Mozes, work that I found beautiful.

Anyway, the pinwheel example brought into play the rotations in the plane in making aperiodic tiling sets, and mathematically, it represents a shift in the subject toward geometry, in the sense of Klein's Erlangen program, geometry viewed through the isometry groups of spaces.

Since the rotation group in three dimensions is much richer than that in two dimensions, it was natural to see what would happen if one started with a substitution there. It was not clear that I could interest John in such a project. Indeed, once my very complicated proof of the pinwheel bumps and dents (more formally called matching rules) was published, John sat me down in his office and informed me that I had wasted a lot of effort and that he would show me how to obtain matching rules for the pinwheel substitution in a very simple way. The idea was to look inside a pinwheel tiling at the neighborhoods of tiles made not just by nearest neighbors, but by next-nearest neighbors, etc., to some appropriate fixed finite depth, and use these to define tiles with matching rules. This technique was, in fact, known to work for some older examples coming from a substitution, and John was sure that it was true generally. So we spent some time on the floor looking at larger and larger neighborhoods in a pinwheel tiling drawn on a large piece of paper. Finally, John let me show him a proof by Ludwig Danzer that this method cannot work for the pinwheel. John was very surprised. And he was now sufficiently appreciative of what I had done on the pinwheel that our discussions became *almost* two-sided.

Also about this time, I told him of an old but unpublished example from 1988 by Peter Schmitt, a single deformed unit cube in space, that was aperiodic. It was based on the following simple idea. One puts bumps and dents on four sides of the cube so it can tile space only if it forms infinite slabs of unit thickness: think of cubes lined up on a table, checkerboard fashion. Then on the tops cut two or three straight grooves (i.e., dents) at an appropriate angle with respect to the edges, and at the bottom corresponding raised bumps (the reverse of the dents), but at an angle with respect to the dents that is irrational with respect to π (see Figure 7).

If you think for a moment, it is clear that this will allow only tilings that consist of infinite parallel slabs rotated irrationally with respect to one another and are therefore aperiodic. Wonderfully simple. Anyway, John was very interested. He had gotten interested in the kite and dart tilings as the smallest known aperiodic set—two tiles—and had spent a lot of time trying to find a single planar tile that was aperiodic, but without success. And here was a simple way to find such a tile in three dimensions! Immediately he wanted to improve it. We spent a few hours (again on the floor, over a large sheet of paper) smoothing out the bumps and dents and came up with what was arguably a nicer example. This illustrates well our different interests in

aperiodicity. For John, it was the challenge of finding simpler and nicer examples of aperiodic tiling sets, and although I tried, I never understood the point.

In 1995, John was willing to consider my proposal to investigate tilings in three dimensions exhibiting special properties. Rather quickly, we (mostly John) arrived at the “quaquaversal” substitution of a triangular prism. The prism, which looks like a door wedge, is shown in Figure 8 decomposed into eight smaller copies of itself. The figure shows two views of the decomposed prism, each fully depicting only four of its components, those illustrated more easily from that vantage. The original prism has five edges of length 1, two edges of length $\sqrt{3}$, and two edges (hypotenuses) of length 2.

It was now harder to “see” what tilings of space are produced by iterating the substitution, but Figure 9 gives some sense of it by omitting many tiles in order to make the structure inside visible. I saw no fundamental reason why

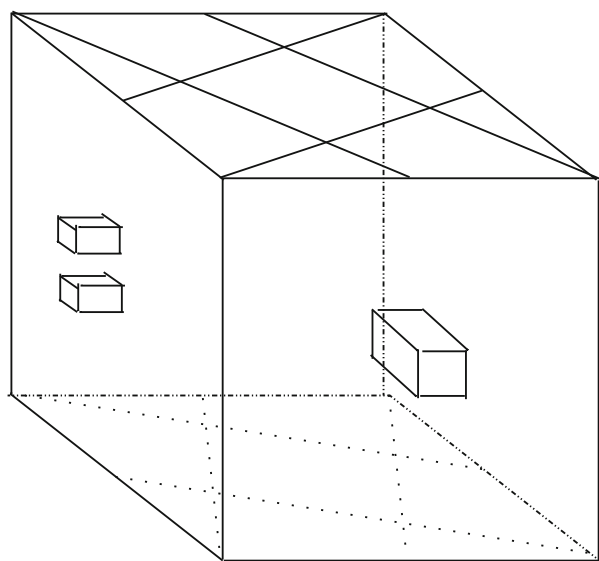


Figure 7. A Schmitt tile.

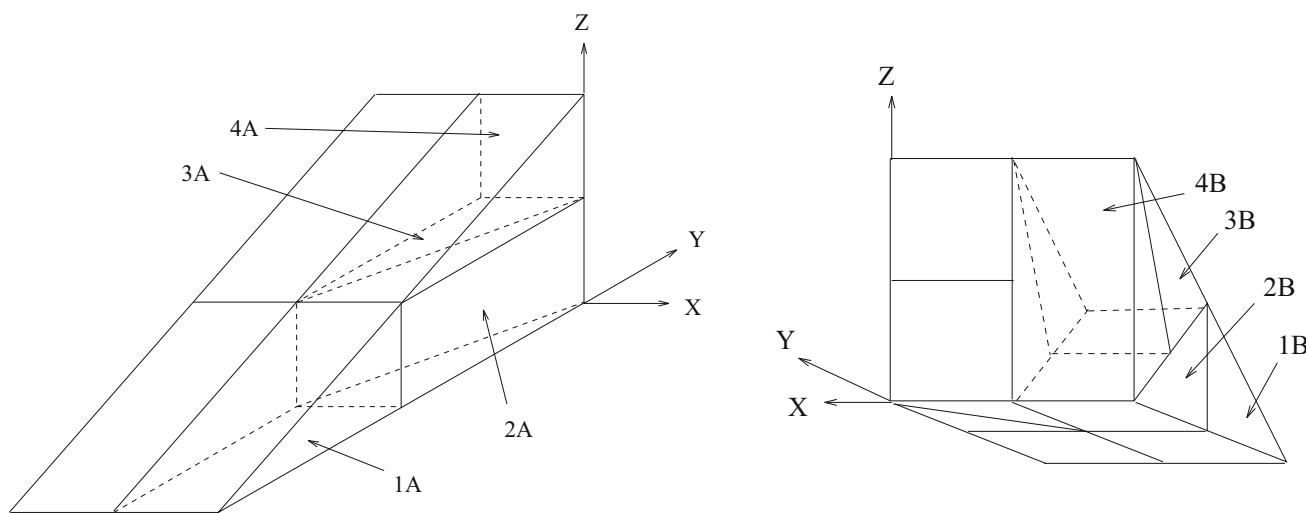


Figure 8. Quaquaversal substitution: two views.

one could not extend the pinwheel technique to produce matching rules for this three-dimensional example, so I never tried, and of course John wasn’t interested in this anyway.

This was our first serious joint project. Our objective was to prove various properties of quaquaversal tilings that were not obtainable in planar tilings such as the pinwheel. For instance, if one looks into a planar substitution tiling, the number of orientations of tiles that appear in an expanding window will grow at most logarithmically with the diameter of the window, as a simple consequence of commutativity of rotations. I hoped that one could get power growth in three dimensions, and in fact, we proved this in [4]. It was not so easy, requiring analyzing the subgroup of the rotation group through which the tile is turned in the substitution process, namely the groups generated by rotation about orthogonal axes, one by $2\pi/3$ and the other by $2\pi/4$. (John liked to represent rotations by quaternions, which was a new world for me.) We called this group $G(3, 4)$ and determined a presentation of it and its subgroup $G(3, 3)$. (Only $G(4, 4)$ and $G(2, n)$ are finite groups.) This led to two more papers (with a third coauthor, Lorenzo Sadun) [5, 6], which studied groups generated when the angle between the rotation axes was more interesting geometrically: “geodetic” angles, whose squared trigonometric functions are rational. (John had disapproved of my choice of the pinwheel name—it seems the toy I based it on is called something else in England—and once we were working together, it was unquestioned that John would produce terms such as “quaquaversal” and “geodetic.” He imported the word quaquaversal from geology, where it refers to structures oriented in all directions.)

Our results have had a number of applications, and in particular, the quaquaversal tiling has lived up to its original motivation. Both the pinwheel tilings and the quaquaversal tilings provide easily computable partitions of (2- and 3-dimensional) space that are statistically round in the sense that the number of orientations of a tiling’s components becomes uniform as the diameter of a window grows into the tiling. Such tilings have therefore been used

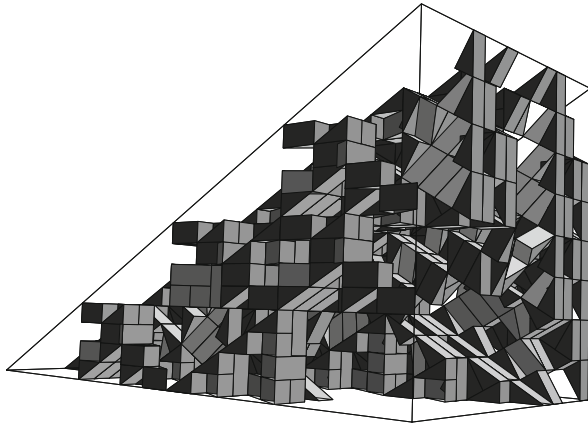


Figure 9. A quaquaversal tiling.

together with numerical solutions of differential equations, and other analyses of discrete data in space, for which one wants to avoid artificial consequences of the symmetries of cubic partitions. In this regard, the pinwheel suffers from the slow (logarithmic) speed at which it exhibits its roundness, and the quaquaversal tilings successfully overcame this weakness. This subject is directly related to expander graphs, and the quaquaversal tilings have recently been analyzed in that context.

Quaquaversal tilings employ successive rotations about orthogonal axes. To connect better with significant polyhedra, in our following papers with Sadun we replaced right angles with geodesic angles and analyzed the subgroups of $SO(3)$ this produced. To understand how we applied this, it is useful to step back a bit.

One way the complexity of the rotation group of three dimensions is exhibited is in fundamental aspects of volume. In two dimensions, a pair of polygons have equal area if and only if each can be decomposed into a common set of polygons, i.e., if they are rearrangements of a common set of simple components. This is the Wallace–Bolyai–Gerwien theorem. But this criterion fails badly in three dimensions (Hilbert’s third problem); two polyhedra are equidecomposable into polyhedra if and only if they have the same volume and have the same Dehn invariant, a result known as Sydler’s theorem. Because of our choice of the class of angles between rotation axes, we were able to apply our results on tiling groups to compute Dehn invariants for Archimedean polyhedra.

But one can go further into the fundamentals of volume. The Banach–Tarski paradox shows that two balls of different volumes can be equidecomposed into a finite number of more complicated (nonmeasurable) components! The intrusion of measurability is significant for our story, as we will see from another interesting aperiodic tiling.

The Binary Tiling of the Hyperbolic Plane

In the same paper in which Penrose wrote up the kite and dart tiling in 1978 [9], he also introduced a tiling of the hyperbolic plane, using congruent copies of a tile with two

sides that are geodesics and two sides that are horocycles. Using the upper-half-plane model of the hyperbolic plane, Figure 10 shows the tiling, and Figure 11 shows the binary tile, which has bumps and dents added.

In any tiling of the plane made from congruent images of a binary tile, its rectangular bumps force tiles to uniquely fill out the strip between two horizontal horocycles, and then the triangular bumps force such strips to uniquely fill out the half-plane below it. There must also be a strip above it, but it is not unique: there are two ways to place it. Therefore, there are uncountably many ways a given binary tile can sit in such a tiling, of which Figure 10 is representative.

Whether the binary tile is “aperiodic” requires a further reinterpretation of the concept, in terms of symmetries of the hyperbolic plane. It is complicated but fortunately unnecessary here. We will just note the connection of these tilings with a famous packing of disks by Károly Böröczky produced a few years earlier, in 1974 [2].

Superimposing a disk in the binary tile yields disk packings such as that depicted in Figure 12. Assume that the “primary” tile has vertices with complex coordinates at i , $2 + i$, $2i$, and $2 + 2i$ and that the disk is centered at $1/2 + 7i/4$. Performing the rigid motion $z \rightarrow 6z/5$ just on the disks results in the primary tile now having disks at $3/5 + 21i/10$ and $3 + 21i/10$, and the full shifted disk packing is as in Figure 13.

A useful notion of density for the disk packing would require that the density be unchanged by the rigid motion, but that would contradict the change from one to two disks in each tile. It took years to come to grips with this amazing example. Thirty years later, an analysis of the situation by Lewis Bowen in his 2002 thesis [3] starting from aperiodicity and using an ergodic theory perspective showed that one can make sense of packing density and aperiodicity in hyperbolic spaces, but one has to avoid pathological or nonmeasurable sets of packings or tilings.

This is getting beyond the scope of this article, so I’ll just sharpen this last point. The aspect of volume elucidated by equidecomposability is basically discrete mathematics in two (Euclidean) dimensions. The Banach–Tarski paradox shows that such a fundamental study of volume becomes much more complicated in three dimensions, involving measurability. On considering volume at a grander scale, namely the relative volumes of asymptotically large sets, one arrives at one of the key goals of discrete geometry: the study of the densest packings of simple bodies (for instance, spheres). And in this realm, Böröczky’s jarring example in the hyperbolic plane eventually forced a form of measurability. In summary, analysis forces its way into this fundamental area of discrete geometry because of the complicated nature of the isometry group of the underlying space.

Penrose introduced two aperiodic tilings in 1978, the kite and dart tilings in the Euclidean plane and the binary tilings of the hyperbolic plane. These tilings introduced geometry into aperiodicity, vastly broadening its appeal, and in particular, it captured John Conway’s interest. New geometric examples then led to reformulations of aperiodicity, which also brought in ergodic theory, which John

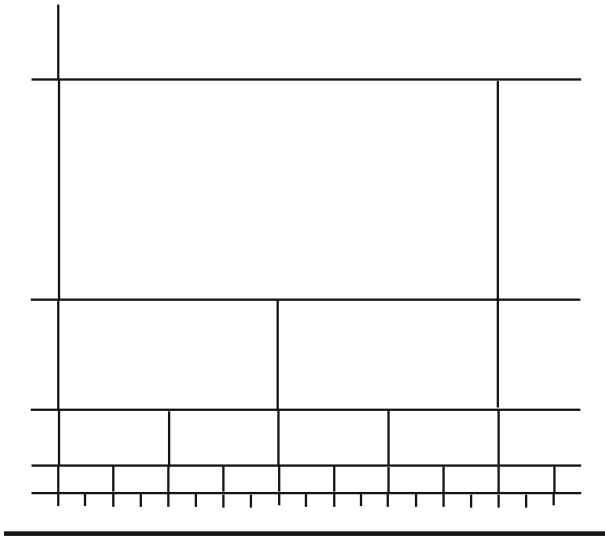


Figure 10. A binary tiling.

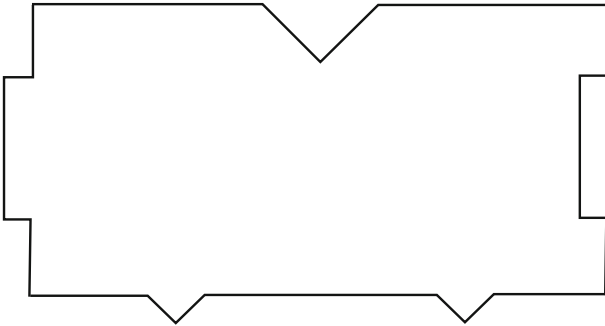


Figure 11. A binary tile.

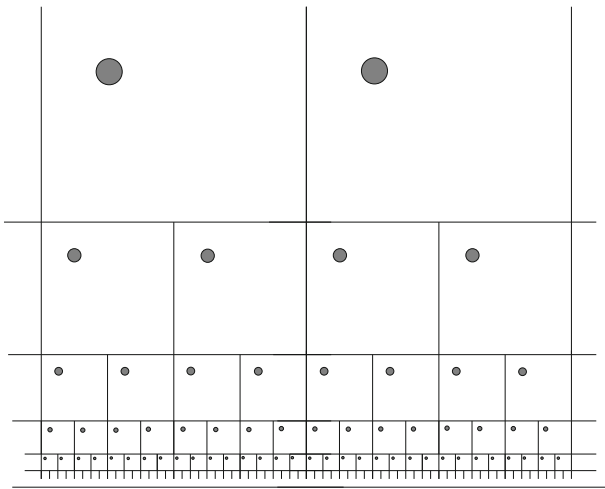


Figure 12. A Böröczky packing.

avoided, but nothing could replace the amazing inventiveness that John brought to the parts of the story that attracted him. This tribute to John is built around provocative geometric examples, a playground in which he was unequalled.

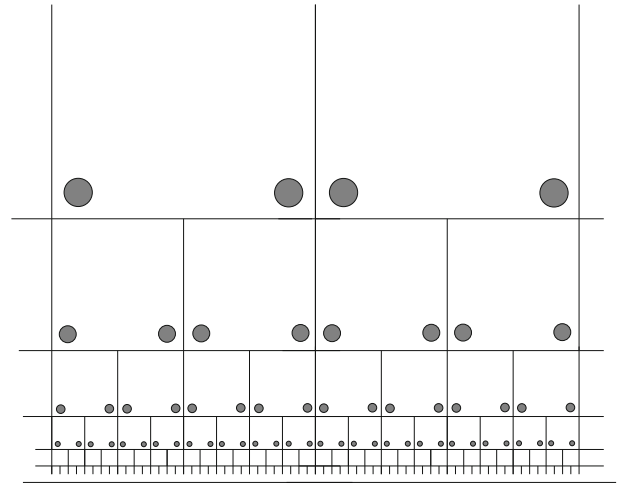


Figure 13. A shifted Böröczky packing.

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